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SUSYQM and other symmetries in quantum mechanics*

G Lévai

Institute of Nuclear Research of the Hungarian Academy of Sciences, PO Box 51, H-4001 Debrecen, Hungary

E-mail: levai@atomki.hu

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Abstract

The relation of supersymmetric quantum mechanics (SUSYQM) is discussed with two other symmetry-based approaches: $\mathcal{PT}$ symmetry and a differential realization of the $su(1,\ 1)$ and $su(2)$ algebra. It is demonstrated that $\mathcal{PT}$ symmetry imposes conditions on the even and odd parts of the real and imaginary components of the superpotential $W(x)$, and these are expressed in terms of a system of first-order linear differential equations, which is homogeneous when the factorization energy is real and inhomogeneous when it is complex. The formal solution of this system is presented for various special cases as well as for the general case. It is shown that a trivial solution of this system corresponds to the unbroken $\mathcal{PT}$ symmetry for the Scarf II potential. The formalism of SUSYQM is also linked with that of the potential algebra approach, and it is demonstrated that the J_+ and J_- ladder operators of some $su(1,\ 1)$ or $su(2)$ algebras act on series of degenerate levels of *different* potentials essentially in the same way as the A and $A^\dagger$ shift operators of SUSYQM. Examples are presented for $su(1,\ 1)$ and $su(2)$ potential algebras, as well as for spectrum generating algebras of the same type. Possible generalizations of this construction are also pointed out.

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1. Introduction

Since its introduction about 20 years ago, supersymmetric quantum mechanics (SUSYQM) has had a strong impact on various branches of physics. Besides its conceptual novelty that offered a new symmetry-based perspective in analysing physical systems, it also stimulated

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the development of the technical machinery of mathematical physics. In most applications of SUSYQM (super)symmetry manifests itself in the degeneracy of the energy spectra of interrelated Hamiltonians. In the simplest and most widely used formulation of this approach, i.e. in $N = 2$ SUSYQM, this clearly follows from a handful of simple relations, but this feature characterizes all other generalizations of this model, including realizations in terms of larger matrices, higher order SUSY operators, iterated SUSY transformations connecting phase-equivalent potentials, multichannel SUSYQM, parafermionic SUSYQM, etc. For reviews on these generalizations and on fundamental physical and mathematical concepts of SUSYQM, we refer to the rich literature on the subject [1].

In the present contribution we focus on $N = 2$ SUSYQM only, and investigate its relation with other symmetry-based approaches, such as $\mathcal{PT}$-symmetric quantum mechanics and some algebraic methods. To exemplify the relation of these symmetry concepts, we consider the one-dimensional Schrödinger equation, with special attention to exactly solvable problems.

We start with recalling the essential formulae of $N = 2$ SUSYQM for later reference. The 'bosonic' and 'fermionic' Hamiltonians forming the diagonal components of the 2×2 matrix SUSY Hamiltonian are factorized as

$$H^{(-)} = A^\dagger A + \varepsilon \qquad H^{(+)} = A A^\dagger + \varepsilon. \tag{1}$$

For the sake of generality we included in (1) a factorization energy ε, which basically represents a shift of the energy scale. Without it the ground-state energy of $H^{(-)}$ (and thus that of the SUSY Hamiltonian) would be zero in the case of unbroken supersymmetry. The isospectrality of $H^{(-)}$ and $H^{(+)}$ follows from (1), irrespective of the realization of the A and $A^\dagger$ operators. In order to construct a one-dimensional Schrödinger operator it is natural to choose these as first-order linear differential operators,

$$A = \frac{\mathrm{d}}{\mathrm{d}x} + W(x) \qquad A^\dagger = -\frac{\mathrm{d}}{\mathrm{d}x} + W(x). \tag{2}$$

The resulting 'bosonic' and 'fermionic' supersymmetric partner potentials are then

$$V^{(-)}(x) = W^2(x) - \frac{\mathrm{d}W}{\mathrm{d}x} + \varepsilon \qquad V^{(+)}(x) = W^2(x) + \frac{\mathrm{d}W}{\mathrm{d}x} + \varepsilon \tag{3}$$

where the $W(x)$ superpotential is related to the ground-state wavefunction of $H^{(-)}$ as

$$W(x) = -\frac{\mathrm{d}}{\mathrm{d}x} \ln \psi_0^{(-)}(x). \tag{4}$$

This latter result follows from the requirement that the energy level corresponding to the ground state of $H^{(-)}$ is missing from the spectrum of $H^{(+)}$. (We do not consider the case of other realizations of $N = 2$ SUSYQM in which the superpotential is obtained from non-physical solutions of the 'bosonic' Schrödinger equation.) The SUSY partner potentials are then related by the formula

$$V_+(x) = V_-(x) - 2\frac{\mathrm{d}^2}{\mathrm{d}x^2} \ln \psi_0^{(-)}(x). \tag{5}$$

In general the functional form of the SUSY partner potentials is different; however, for many well-known potentials it is the same, with only some parameters differing in them. This condition defines shape-invariant potentials [2] via the relation

$$V_+(x; a_0) - V_-(x; a_1) \equiv W^2(x; a_0) + W'(x; a_0) - W^2(x; a_1) + W'(x; a_1) = R(a_1). \tag{6}$$

The parameters a_0 and a_1 are connected by some simple (additive) function. Obviously, once the functional form of the SUSY partner potentials is the same, the SUSY transformation equation can be iterated and a whole sequence of potentials can be generated. These will be

isospectral by construction, except that the lowest level is removed in each step. There are altogether 12 shape-invariant potentials [3] (although some of them are mathematically the same and they are treated as separate entities only for historical reasons) and they include the most well-known textbook examples, such as the harmonic oscillator, Coulomb and Morse potentials, together with various variants of the Pöschl–Teller, Scarf and Rosen–Morse potentials. These potentials are easy to handle mathematically, so they represent ideal subjects for analysing various symmetry concepts appearing in quantum mechanics. These potentials have been classified following different principles based on the factorization method [4], algebraic realizations [5], spectral structure [6] and variable transformations [3], but the outcome was essentially the same in each case.

In what follows (in section 2) we first present a simple method to derive solvable (including shape-invariant) potentials, then analyse the relation of SUSYQM with $\mathcal{PT}$ symmetry (in section 3) and with some algebraic approaches (in section 4). Finally, a summary is given (in section 5).

2. Solvable potentials from variable transformations

In this section we first present a simple transformation procedure [7] by which a rather general class of exactly solvable potentials can be derived in a straightforward way. A generalization of this method has been given in [8], where exact (but partly implicit) formulae have been presented for the solution of the six-parameter Natanzon-class potentials. The common feature of these potentials is that their solutions can be written in terms of hypergeometric or confluent hypergeometric functions. The method we outline here is suitable to solve the Schrödinger equation

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + (E - V(x))\psi(x) = 0 \tag{7}$$

for such potentials. For this one assumes that the solutions are written as $f(x)F(z(x))$, where $F(z)$ is some special function of mathematical physics that satisfies the second-order differential equation

$$\frac{\mathrm{d}^2F}{\mathrm{d}z^2} + Q(z)\frac{\mathrm{d}F}{\mathrm{d}z} + R(z)F(z) = 0. \tag{8}$$

The function $z(x)$ then defines a variable transformation. Straightforward calculation shows then that $E - V(x)$ takes the form

$$E - V(x) = \frac{z'''(x)}{2z'(x)} - \frac{3}{4}\left(\frac{z''(x)}{z'(x)}\right)^2 + (z'(x))^2\left(R(z(x)) - \frac{1}{2}\frac{\mathrm{d}Q}{\mathrm{d}z} - \frac{1}{4}Q^2(z(x))\right) \tag{9}$$

and the solutions can be written as

$$\psi(x) \sim (z'(x))^{-\frac{1}{2}} \exp\left(\frac{1}{2}\int^{z(x)} Q(z)\,\mathrm{d}z\right) F(z(x)). \tag{10}$$

Note that besides the $R(z)$ and $Q(z)$ functions (which are known after we have chosen the special function $F(z)$) equations (9) and (10) depend only on the $z(x)$ function.

The next step is choosing $z(x)$ in a meaningful way, i.e. such that it leads to a solvable potential. For this we select on the right-hand side of (10) some term (or combination of terms) that includes $(z'(x))^2$ and equate it with a constant in order to account for E on the left-hand side. This leads to a differential equation

$$\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2 \phi(z) = C \tag{11}$$

where $\phi(x)$ is a function originating from $Q(z)$ and $R(z)$. The $z(x)$ function can then be determined by direct integration. The only remaining task is to redefine the potential parameters such that the n principal quantum number appears only in the constant (energy) term. Sometimes only the implicit $x(z)$ function can be determined from (11), but the 'implicit' potentials derived this way are also exactly solvable, as any relevant quantity can be calculated for them to any desired accuracy.

The most well-known potentials, the members of the shape-invariant class [2], are obtained by considering a single term for $\phi(z)$ in (11), while the general Natanzon-class potentials are obtained by taking a two- or three-term linear combination for $\phi(z)$. The resulting formulae are relatively simple for shape-invariant potentials, while they might become impossible to handle when a three-term expression is taken. The most well-known non-shape-invariant Natanzon-class potentials (e.g., the Ginocchio [9, 10], the generalized Coulomb [11, 12] and some others [13, 14]) have been obtained by considering a two-term combination in $\phi(z)$. Obviously, most of these can be reduced to some shape-invariant potentials by tuning some parameters to special values (e.g., to zero).

Finally, we note that equation (9) can be cast in a form familiar to SUSYQM:

$$E - V(x) = (z'(x))^2 R(z(x)) - \left[\left(\frac{f'(x)}{f(x)}\right)^2 + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f'(x)}{f(x)}\right)\right]. \tag{12}$$

We may notice that whenever $R(z)$ vanishes for the ground state $n = 0$ (this is the case when $F(z)$ is an orthogonal polynomial, a special case of the hypergeometric or confluent hypergeometric function), a superpotential can be defined as

$$W(x) = -\frac{\mathrm{d}}{\mathrm{d}x}\ln f(x) = -\frac{1}{2}Q(z(x))z'(x) + \frac{1}{2}\frac{z''(x)}{z'(x)}. \tag{13}$$

3. SUSY and $\mathcal{PT}$ symmetry

In quantum mechanics $\mathcal{PT}$ symmetry requires the invariance of a potential under the simultaneous action of the $\mathcal{P}$ spatial and $\mathcal{T}$ time reflection operations (the latter essentially being complex conjugation). For one-dimensional potentials of non-relativistic quantum mechanics this invariance requires $V^*(-x) = V(x)$, therefore the real and imaginary component of a $\mathcal{PT}$-invariant potential has to be an even and odd function of x, respectively. These potentials represent a rather peculiar class among complex potentials, since it was found that their discrete energy eigenvalues can be *real* [15]. This is different from complex potentials appearing, e.g., in nuclear physics imitating the absorption of particles in a nuclear reaction, since the discrete energy eigenvalues were found to be complex in that case.

It soon turned out that $\mathcal{PT}$ symmetry is neither a necessary nor a sufficient condition for having real energy eigenvalues in a complex potential. A typical feature appearing in most $\mathcal{PT}$-symmetric potentials was the pairwise merging of real energy eigenvalues and their re-emergence as complex conjugate pairs as some potential parameter was tuned. At the same time the corresponding solutions ceased to be eigenfunctions of the $\mathcal{PT}$ operator, so this phenomenon was interpreted as the spontaneous breakdown of $\mathcal{PT}$ symmetry [15].

Besides the real energy eigenvalues there were further signs indicating that $\mathcal{PT}$-symmetric potentials share some features with Hermitian problems. For example, with the modification of the inner product the orthogonality of the energy eigenstates could be restored, and also a modified continuity equation could be derived. However, the price to be paid was that the pseudo-norm defined this way had an indefinite sign, questioning the probabilistic interpretation of the wavefunctions. These unusual results have finally been interpreted in

terms of pseudo-Hermitian Hamiltonians and anti-linear operators [16], as $\mathcal{PT}$ symmetry represents a special case of these. With further modification of the inner product the positive norm could also be restored [17].

Strangely enough the first examples of $\mathcal{PT}$-symmetric potentials have been derived in numerical studies, but soon the $\mathcal{PT}$-symmetric versions of solvable potentials have also been constructed. In a systematic study the $\mathcal{PT}$-symmetric versions of shape-invariant potentials have been constructed and conditions have been formulated for having real [18] or complex [19] energy eigenvalues in their spectra, i.e. for having unbroken or spontaneously broken $\mathcal{PT}$ symmetry. These results followed from adapting the methods discussed in section 2 to the $\mathcal{PT}$-symmetric setting.

An unusual feature characterizing these solvable potentials was the appearance of a *second* set of solutions with the same principal quantum number n. These solutions could be identified using the $q = \pm 1$ quasi-parity quantum number [20], which turns into conventional parity when the $\mathcal{PT}$-symmetric harmonic oscillator is reduced to its Hermitian version. It was also found that the second set of solutions appears due to the less strict boundary conditions. In some potentials these can be implemented by shifting *formally* the potential in the imaginary direction via $x \to x + \mathrm{i}\epsilon$ and thus eliminating the singularities, e.g., at $x = 0$. Here ϵ appears as a constant of integration from (11).

Supersymmetric quantum mechanics has been combined with $\mathcal{PT}$ symmetry in various other ways. Fundamental mathematical aspects have been discussed in [21, 16]; the reality of the spectrum of $\mathcal{PT}$-symmetric SUSY partners of real potentials has been pointed out [22]; realizations in terms of second-order SUSYQM and parasupersymmetric quantum mechanics have been constructed [23] and a quasi-parity-dependent factorization energy has been introduced in order to account for the dual structure of the energy levels [24].

Here our aim is to analyse how the complex structure of the $W(x)$ superpotential influences the $\mathcal{PT}$ symmetry of the SUSY partner potentials (3). For this we separate the superpotential into real and imaginary components and then split both of them into even and odd functions of x:

$$W(x) = W_{\mathrm{R}}(x) + \mathrm{i}W_{\mathrm{I}}(x) = W_{\mathrm{Re}}(x) + W_{\mathrm{Ro}}(x) + \mathrm{i}W_{\mathrm{Ie}}(x) + \mathrm{i}W_{\mathrm{Io}}(x). \tag{14}$$

Separating the $V^{(-)}(x)$ potential in a similar fashion one gets

$$V^{(-)}(x) = V^{(-)}_{\mathrm{Re}}(x) + V^{(-)}_{\mathrm{Ro}}(x) + \mathrm{i}V^{(-)}_{\mathrm{Ie}}(x) + \mathrm{i}V^{(-)}_{\mathrm{Io}}(x) \tag{15}$$

with

$$\begin{aligned}
V^{(-)}_{\mathrm{Re}}(x) &= W^2_{\mathrm{Re}}(x) + W^2_{\mathrm{Ro}}(x) - W^2_{\mathrm{Ie}}(x) - W^2_{\mathrm{Io}}(x) - W'_{\mathrm{Ro}}(x) + \mathrm{Re}(\varepsilon) \\
V^{(-)}_{\mathrm{Ro}}(x) &= 2W_{\mathrm{Re}}(x)W_{\mathrm{Ro}}(x) - 2W_{\mathrm{Ie}}(x)W_{\mathrm{Io}}(x) - W'_{\mathrm{Re}}(x) \\
V^{(-)}_{\mathrm{Ie}}(x) &= 2W_{\mathrm{Re}}(x)W_{\mathrm{Ie}}(x) + 2W_{\mathrm{Ro}}(x)W_{\mathrm{Io}}(x) - W'_{\mathrm{Io}}(x) + \mathrm{Im}(\varepsilon) \\
V^{(-)}_{\mathrm{Io}}(x) &= 2W_{\mathrm{Re}}(x)W_{\mathrm{Io}}(x) + 2W_{\mathrm{Ro}}(x)W_{\mathrm{Ie}}(x) - W'_{\mathrm{Ie}}(x).
\end{aligned} \tag{16}$$

The $\mathcal{PT}$ symmetry requirement $V^*(-x) = V(x)$ means for $V^{(-)}(x)$ that $\mathrm{Re}(V^{(-)}(x))$ has to be an even function of x, while $\mathrm{Im}(V^{(-)}(x))$ has to be odd, and this leads to two coupled first-order differential equations:

$$\begin{aligned}
W'_{\mathrm{Re}}(x) - 2W_{\mathrm{Ro}}(x)W_{\mathrm{Re}}(x) + 2W_{\mathrm{Ie}}(x)W_{\mathrm{Io}}(x) &= 0 \\
W'_{\mathrm{Io}}(x) - 2W_{\mathrm{Re}}(x)W_{\mathrm{Ie}}(x) - 2W_{\mathrm{Ro}}(x)W_{\mathrm{Io}}(x) &= \mathrm{Im}(\varepsilon).
\end{aligned} \tag{17}$$

This can be considered as a system of inhomogeneous (or nonhomogeneous) linear first-order differential equations for the functions $W_{\mathrm{Re}}(x)$ and $W_{\mathrm{Io}}(x)$, where the inhomogeneity is represented by a constant in only one of the equations; furthermore, the coefficients appearing

in the two equations are expressed in terms of the same two functions, $W_{\mathrm{Ro}}(x)$ and $W_{\mathrm{Ie}}(x)$. Before discussing the general problem of solving (17), we first derive formulae analogous to (16) for the SUSY partner $V^{(+)}(x)$:

$$\begin{aligned}
V^{(+)}_{\mathrm{Re}}(x) &= W^2_{\mathrm{Re}}(x) + W^2_{\mathrm{Ro}}(x) - W^2_{\mathrm{Ie}}(x) - W^2_{\mathrm{Io}}(x) + W'_{\mathrm{Ro}}(x) + \mathrm{Re}(\varepsilon) \\
V^{(+)}_{\mathrm{Ro}}(x) &= 4W_{\mathrm{Re}}(x)W_{\mathrm{Ro}}(x) - 4W_{\mathrm{Ie}}(x)W_{\mathrm{Io}}(x) \\
V^{(+)}_{\mathrm{Ie}}(x) &= 4W_{\mathrm{Re}}(x)W_{\mathrm{Ie}}(x) + 4W_{\mathrm{Ro}}(x)W_{\mathrm{Io}}(x) + 2\mathrm{Im}(\varepsilon) \\
V^{(+)}_{\mathrm{Io}}(x) &= 2W_{\mathrm{Re}}(x)W_{\mathrm{Io}}(x) + 2W_{\mathrm{Ro}}(x)W_{\mathrm{Ie}}(x) + W'_{\mathrm{Ie}}(x).
\end{aligned} \tag{18}$$

Here we eliminated the derivatives from $V^{(+)}_{\mathrm{Ro}}(x)$ and $V^{(+)}_{\mathrm{Ie}}(x)$ using (17). Let us now discuss the solution of (17) for increasingly complex situations and its implication on the $\mathcal{PT}$ symmetry of $V^{(+)}(x)$.

(a) $W_{\mathrm{Re}}(x) = W_{\mathrm{Io}}(x) = 0$. This case represents a trivial solution, and it necessarily leads to $\mathrm{Im}(\varepsilon) = 0$, reducing (17) to a homogeneous system of differential equations. It is straightforward to prove that in this case $V^{(+)}(x)$ is also $\mathcal{PT}$-symmetric.

(b) $W_{\mathrm{Io}}(x) = 0$, $W_{\mathrm{Re}}(x) \neq 0$. In this case the integration of (17) is a straightforward task, leading to

$$W_{\mathrm{Re}}(x) = C\exp\left(2\int_0^x W_{\mathrm{Ro}}(x')\,\mathrm{d}x'\right) \qquad W_{\mathrm{Ie}}(x) = -\frac{\mathrm{Im}(\varepsilon)}{2}W^{-1}_{\mathrm{Re}}(x) \tag{19}$$

where we have already taken into account through the boundary conditions the parity requirement for $W_{\mathrm{Re}}(x)$. It can be proved that in this case the imaginary component of $V^{(+)}(x)$ is odd; however, its real component does not have definite parity, therefore the SUSY partner is not $\mathcal{PT}$-symmetric.

(c) $W_{\mathrm{Re}}(x) = 0$, $W_{\mathrm{Io}}(x) \neq 0$. Here the solution of (17) becomes

$$\begin{aligned}
W_{\mathrm{Io}}(x) &= \mathrm{Im}(\varepsilon)\exp\left(2\int_0^x W_{\mathrm{Ro}}(x')\,\mathrm{d}x'\right)\int_0^x\left[\exp\left(-2\int_0^{x'} W_{\mathrm{Ro}}(x'')\,\mathrm{d}x''\right)\right]\mathrm{d}x' \\
W_{\mathrm{Ie}}(x) &= 0
\end{aligned} \tag{20}$$

where we have again considered appropriate boundary conditions rendering $W_{\mathrm{Io}}(x)$ to an odd function of x. In this case $V^{(-)}(x)$ is an even real function (a special case of $\mathcal{PT}$ symmetry), while its SUSY partner, $V^{(+)}(x)$, is an even imaginary function, so it cannot be $\mathcal{PT}$-symmetric.

(d) *The general case.* The general solution of (17) can be given in the following form [25]:

$$\begin{aligned}
W_{\mathrm{Re}}(x) &= \exp\left(2\int_0^x W_{\mathrm{Ro}}(x')\,\mathrm{d}x'\right)\cos\left(-2\int_0^x W_{\mathrm{Ie}}(x')\,\mathrm{d}x'\right) \\
&\quad + \mathrm{Im}(\varepsilon)\int_0^x \exp\left(2\int_s^x W_{\mathrm{Ro}}(x')\,\mathrm{d}x'\right)\sin\left(-2\int_s^x W_{\mathrm{Ie}}(x')\,\mathrm{d}x'\right)\mathrm{d}s \\
W_{\mathrm{Io}}(x) &= -\exp\left(2\int_0^x W_{\mathrm{Ro}}(x')\,\mathrm{d}x'\right)\sin\left(-2\int_0^x W_{\mathrm{Ie}}(x')\,\mathrm{d}x'\right) \\
&\quad + \mathrm{Im}(\varepsilon)\int_0^x \exp\left(2\int_s^x W_{\mathrm{Ro}}(x')\,\mathrm{d}x'\right)\cos\left(-2\int_s^x W_{\mathrm{Ie}}(x')\,\mathrm{d}x'\right)\mathrm{d}s.
\end{aligned} \tag{21}$$

Now $W_{\mathrm{Re}}(x)$ is indeed an even function of x, while $W_{\mathrm{Io}}(x)$ is odd. The SUSY partner $V^{(+)}(x)$ does not have $\mathcal{PT}$ symmetry in this case in general.

Note that taking a real factorization energy ε which turns (17) into a homogeneous system reduces the complexity of the solutions in all four cases. $\mathrm{Im}(\varepsilon)$ typically appears in the imaginary components of the superpotential. In case (d) it also appears in $W_{\mathrm{Re}}(x)$, and when it is zero, $W_{\mathrm{Re}}(x)$ and $W_{\mathrm{Io}}(x)$ become proportional to each other:

$$W_{\mathrm{Re}}(x) = t(x)W_{\mathrm{Io}}(x) \tag{22}$$

$$t(x) = \cot\left(2\int^{x} W_{\mathrm{Ie}}(x')\,\mathrm{d}x'\right). \tag{23}$$

This $t(x)$ function can also be obtained in a different way, when one derives a particular solution of the homogeneous version of (17) by substituting (22) into it.

Although the formal solution of (17) seems a simple task, one might face a number of technical difficulties in evaluating the formulae for concrete examples. So here we present an example in which the validity of (17) is proved directly. The $\mathcal{PT}$-symmetric Scarf II potential (which is independent of q due to $q^2 = 1$) and its SUSY partner are [24]

$$\begin{aligned} V^{(-)}(x) &= -\frac{1}{\cosh^2 x}\left[\left(\frac{q\alpha+\beta}{2}\right)^2 + \left(\frac{q\alpha-\beta}{2}\right)^2 - \frac{1}{4}\right] + \frac{2\mathrm{i}\sinh x}{\cosh^2 x}\left(\frac{\beta+q\alpha}{2}\right)\left(\frac{\beta-q\alpha}{2}\right) \\ V_q^{(+)}(x) &= -\frac{1}{\cosh^2 x}\left[\left(\frac{q\alpha+\beta+2}{2}\right)^2 + \left(\frac{q\alpha-\beta}{2}\right)^2 - \frac{1}{4}\right] \\ &\quad + \frac{2\mathrm{i}\sinh x}{\cosh^2 x}\left(\frac{\beta+q\alpha+2}{2}\right)\left(\frac{\beta-q\alpha}{2}\right) \end{aligned} \tag{24}$$

where we have used the factorization energy $\varepsilon = \varepsilon_q = -(q\alpha+\beta+1)^2/4$, while the corresponding superpotential is written as

$$W^{(q)}(x) = -\frac{1}{2}(q\alpha+\beta+1)\tanh x - \frac{\mathrm{i}}{2}(\beta-q\alpha)\frac{1}{\cosh x}. \tag{25}$$

We find that for real α, $W_{\mathrm{Re}}(x) = W_{\mathrm{Io}}(x) = 0$ holds, and the factorization energy is real, corresponding to the unbroken $\mathcal{PT}$ symmetry of both $V^{(-)}(x)$ and $V_q^{(+)}(x)$. Furthermore, when α is imaginary, we get the inhomogeneous system of equations (17) with

$$\begin{aligned} W_{\mathrm{Re}}(x) &= \frac{\mathrm{i}q\alpha}{2}\frac{1}{\cosh x} & W_{\mathrm{Ro}}(x) &= -\frac{\beta+1}{2}\tanh x \\ W_{\mathrm{Ie}}(x) &= -\frac{\beta}{2\cosh x} & W_{\mathrm{Io}}(x) &= \frac{\mathrm{i}q\alpha}{2}\tanh x. \end{aligned} \tag{26}$$

In this case, which corresponds to the breakdown of $\mathcal{PT}$ symmetry (spontaneous for $V^{(-)}(x)$ and manifest for $V_q^{(+)}(x)$), the imaginary component of the factorization energy appearing in (17) is $\mathrm{Im}(\varepsilon) = \mathrm{i}q\alpha(\beta+1)/2$ in agreement with previous results [24].

We note here that the $\mathcal{PT}$-symmetric Scarf II potential can also be written in a more general form by using the imaginary coordinate shift $x \to x + \mathrm{i}\epsilon$, but this would not change the conclusions, and only would make the separation of the real and imaginary components more involved technically.

We have seen that for the most well-known $\mathcal{PT}$-symmetric potentials unbroken $\mathcal{PT}$ symmetry occurs for the trivial $W_{\mathrm{Re}}(x) = W_{\mathrm{Io}}(x) = 0$ solutions of (17), which reduce to a homogeneous system of differential equations. The formulae presented above, however, allow more general construction of $W(x)$ superpotentials leading to a $\mathcal{PT}$-symmetric $V^{(-)}(x)$ potential, and a generally non-$\mathcal{PT}$-symmetric SUSY partner. One interesting question is whether there are any examples for which a non-trivial solution of the homogeneous equation

is obtained (this is the case of a real factorization energy), and also what one can find out about the $\mathcal{PT}$ symmetry of $V^{(+)}(x)$.

Another important point is that the general construction of solving equation (17) works for analytically and numerically solvable potentials too. So it is possible to generate potentials which are $\mathcal{PT}$-symmetric by construction by selecting some $W_{\mathrm{Ro}}(x)$ and $W_{\mathrm{Ie}}(x)$ functions and deriving from them the full superpotential. It would then be an interesting task to investigate situations in which the $\mathcal{PT}$ symmetry of the $V^{(-)}(x)$ potential is intact or is spontaneously broken, and also to study how this affects the $\mathcal{PT}$ symmetry of the SUSY partners. This would be an interesting task even if the solutions cannot be determined in an explicit form (apart from the ground state of $V^{(-)}(x)$, which is determined by the $W(x)$ superpotential).

4. SUSYQM and algebraic approaches

Algebraic constructions have been used in the analysis of the Schrödinger equation and its solutions since the formulation of quantum mechanics. In contrast with SUSYQM-based methods, which can equally be applied to exactly and numerically solvable problems, algebraic approaches generally focus on exactly solvable problems, as closed analytic expressions are essential to formulate the machinery of group theory.

A common feature of algebraic approaches is that the eigenstates of some potentials serve as a basis for the irreducible representation of certain groups, so the generators of this group ladder between the solutions. Depending on the nature of these connections one can talk about symmetry (or degeneracy), spectrum generating, dynamical or potential algebras, for example [26, 27]. In the first case the ladder operators connect degenerate levels, and the Hamiltonian commutes with the elements of the algebra, so it has a symmetry defined by the corresponding group. In the second case the elements of the algebra ladder along eigenstates belonging to different energies, so they 'generate' the energy spectrum. In some cases all the states of a system can be accommodated in a single irreducible representation of some group, which contains the symmetry and spectrum generating group as subgroups. This defines a dynamical group. It has to be mentioned though that in one-dimensional quantum mechanical potentials the degeneracy of levels cannot happen, except when one considers a radial problem in which eigenstates can occur with the same energy but with different orbital angular momentum l. This is the case with the (three-dimensional) Coulomb and harmonic oscillator potentials, which have $so(4, 2)$ [28] and $mp(6)$ [29] as a dynamical group, respectively. Potential algebras are somewhat similar to symmetry algebras in the sense that their elements connect eigenstates with the same energy; however, these states are eigenstates of *different* potentials, i.e. potentials of the same type, but with different coupling coefficients. Depending on whether the series of degenerate levels in question is finite or infinite, the potential algebra is compact or non-compact, since the discrete unitary irreducible representations of the corresponding groups are finite and infinite dimensional in the two cases, respectively. Non-compact potential algebras also have continuous unitary irreducible representations, which can be associated with scattering states [27].

From the possible algebraic approaches here we consider a particular differential realization of the $su(1, 1)$ algebra (and its compact counterpart $su(2)$), because this can be related to the formalism of SUSYQM. Let us consider the $su(1, 1) \sim so(2, 1)$ algebra defined by the commutation relations

$$[J_z, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = -2J_z \tag{27}$$

and construct the generators as first-order differential operators of the type

$$J_\pm = \mathrm{e}^{\pm \mathrm{i}\phi}\left(\pm h(x)\frac{\partial}{\partial x} \pm g(x) + k(x)J_z + c(x)\right) \qquad J_z = -\mathrm{i}\frac{\partial}{\partial \phi} \tag{28}$$

where the four functions appearing in (28) will be defined later. Let us also write the basis states in the form

$$\langle x|jm\rangle = \Psi_{jm}(x,\phi) = \mathrm{e}^{\mathrm{i}m\phi}\psi_{jm}(x) \tag{29}$$

where $\psi_{jm}(x)$ is the physical solution of a Schrödinger equation, ϕ is an auxiliary phase variable, while j and m are the labels of the irreducible representations of the corresponding group $SU(1,1)$. These latter indices have to satisfy the eigenvalue equation for the Casimir operator and J_z,

$$C_2|jm\rangle \equiv \left(-J_+J_- + J_z^2 - J_z\right)|jm\rangle = j(j+1)|jm\rangle \tag{30}$$

$$J_z|jm\rangle = m|jm\rangle. \tag{31}$$

In order to satisfy the commutation relations (27) the following two conditions have to be fulfilled by three of the four undefined functions appearing in (28):

$$k^2(x) - h(x)\frac{\mathrm{d}k}{\mathrm{d}x} = 1 \qquad c(x)k(x) - h(x)\frac{\mathrm{d}c}{\mathrm{d}x} = 0. \tag{32}$$

Equation (30) is a second-order differential equation, and our aim is to associate it with the one-dimensional Schrödinger equation. For this the linear derivative term has to be eliminated, which leads to a third relation between the functions appearing in (28):

$$g(x) = \frac{1}{2}\left(k(x) - \frac{\mathrm{d}h}{\mathrm{d}x}\right). \tag{33}$$

The eigenvalue equation (30) is then similar to a Schrödinger equation:

$$\begin{aligned}[C_2 - j(j+1)]|jm\rangle = -h^2\Bigg[&-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \left(\frac{h'}{2h}\right)^2 + \left(\frac{k}{2h}\right)^2 - \frac{k'}{2h} + \frac{h''}{2h} \\ &+ \frac{c^2}{h^2} + \frac{2ck}{h^2}m + \frac{1-k^2}{h^2}m^2 + \frac{j(j+1)}{h^2}\Bigg]|jm\rangle = 0.\end{aligned} \tag{34}$$

This equation contains the j and m labels, which will appear in the potential parameters and also in the energy expression. Imposing the three conditions in (32) and (33) on the four functions leaves us with only one independent function, and we can choose this to be $h(x)$ without loss of generality.

We may notice that in deriving (34) from (30) we essentially proceeded along the same route as in section 2, where we started with the second-order differential equation (8), performed variable and similarity transformations and finally arrived at the Schrödinger equation (7) that contains only one unknown function. The variable transformation (controlled in section 2 by $z(x)$) can be defined in this case as

$$h(x) \to h(z) = h(x(z))\frac{\mathrm{d}z}{\mathrm{d}x} \qquad \psi_{jm}(x) \to \psi_{jm}(x(z)) \tag{35}$$

while the similarity transformation (accounted for by $f(x)$ in section 2) is

$$g(x) \to g(x) + h(x)\frac{\mathrm{d}}{\mathrm{d}x}\ln v(x) \qquad \psi_{jm}(x) \to v^{-1}(x)\psi_{jm}(x). \tag{36}$$

This construction has been applied to the shape-invariant potentials [30] and it was shown that most of them accommodate a spectrum generating algebra, a potential algebra or both.

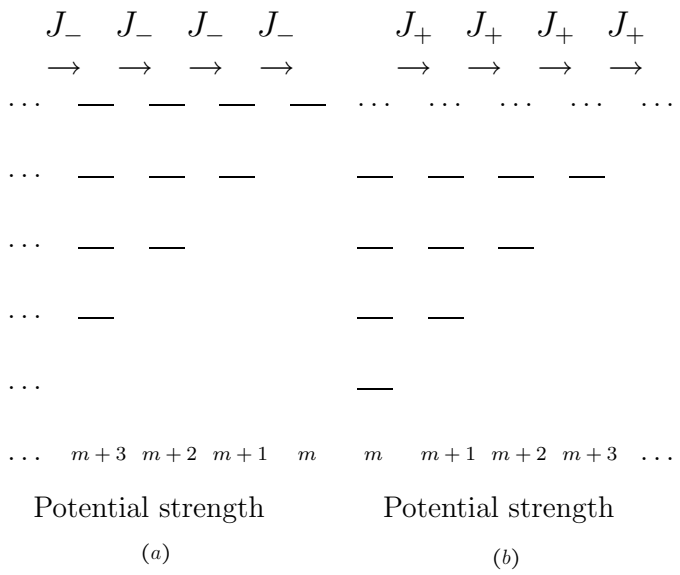

Figure 1. Schematic illustration of the bound-state spectrum in the case of a non-compact *su*(1, 1) (panel *a*) and a compact *su*(2) (panel *b*) potential algebra. In the former case there is an infinite series of degenerate levels belonging to different values of m, defining different potential strengths, but all the potentials have a finite number of bound states. In the latter case the number of bound states is infinite in each potential, but there is only a finite number of levels belonging to each energy. The energy eigenvalues depend on the $j(j+1)$ eigenvalue of the Casimir invariant.

Exceptions were potentials that have expressions of the type $(n+c)^{-2}$ in their bound-state energy formula. This was the case with the Coulomb (type LII in the notation of [3], and F in [4]) potential and the general case of the Rosen–Morse and Eckart potentials (belonging to the PII [3] or E [4] type). In all other cases, $n = j+m$ appeared in linear or quadratic expressions as expected from (34).

The potential and spectrum generating algebras appeared either as the compact *su*(1, 1) or as the non-compact *su*(2). It is easy to see from the construction that a transition between these two algebras can be generated by considering the transformation

$$[h, g, k, c] \leftrightarrow [\mathrm{i}h, \mathrm{i}g, \mathrm{i}k, \mathrm{i}c]. \tag{37}$$

Potential algebras have been recovered in [30] for two classes of shape-invariant potentials: the LIII class (i.e. the Morse potential) and the PI class which contains five individual potentials. A characteristic feature of these potentials is that the $h(x)$ function in (28) is a constant, and the differential form of the Casimir operator (34) is proportional to the Schrödinger equation up to a constant. For the same reason the J_+ and J_- generators of these potential algebras are practically identical with the $A^\dagger$ and A ladder operators of SUSYQM in the sense that they have the same effect on the solutions [30]. This was confirmed later also in [31, 32]. These potentials belonging to the PI and LIII shape-invariant class correspond to type A and B potentials in the factorization method [4] and a study based on the Lie theory of special functions [5]. See also [3] for the details.

We illustrate the situation in figure 1, where the case of an *su*(1, 1) and an *su*(2) potential algebra is presented schematically. As concrete examples we consider the case of the symmetric Pöschl–Teller potential, which appears as the special case of several PI and

PII class potentials. Taking

$$J_{\pm} = \mathrm{e}^{\pm \mathrm{i}\phi}\left[\pm\frac{\partial}{\partial x} - \tanh x\left(J_z \pm \frac{1}{2}\right)\right] \tag{38}$$

we obtain $su(1,1)$ generators laddering along the degenerate levels of an infinite sequence of potentials $V_m(x) = -\left(m^2 - \frac{1}{4}\right)\cosh^{-2} x$ as on the left panel of figure 1. The energies depend on the j representation label as $E_j = -(j+\frac{1}{2})^2$. The situation on the right panel can be obtained by taking the (37) transformation, which changes (38) to

$$J_{\pm} = \mathrm{e}^{\pm \mathrm{i}\phi}\left[\mp\frac{\partial}{\partial y} - \tan y\left(J_z \pm \frac{1}{2}\right)\right] \tag{39}$$

which also means replacing x with $y = \mathrm{i}x$ as the coordinate. This transformation changes the potential to $V_{\mathrm{m}}(y) = \left(m^2 - \frac{1}{4}\right)\cos^{-2} y$ and the energies to $E_j = \left(j + \frac{1}{2}\right)^2$.

With the $h(x) \neq$ const choice other types of algebras can be obtained, among them spectrum generating ones for the LI, HI classes (i.e. the harmonic oscillators in three and one dimensions) and for special (symmetric) cases of PI and PII potentials. In this case $\langle C_2\rangle = j(j+1)$ will be related to the potential strength and $\langle J_z\rangle = m$ will appear in the energy eigenvalues. An example is the symmetric Pöschl–Teller potential, which is now written in the form $V_j(x) = -j(j+1)\cosh^{-2} x$ with energy eigenvalues $E_m = -m^2$, which is obtained using the $su(2)$ operators,

$$J_{\pm} = \mathrm{i}\mathrm{e}^{\pm \mathrm{i}\phi}\cosh x\left(\pm\frac{\partial}{\partial x} + \tanh x\, J_z\right). \tag{40}$$

In contrast with (38) these $su(2)$ operators ladder along the finite number levels forming finite 'towers' on the left panel of figure 1. A transformation to the trigonometric version of this potential and to an $su(1,1)$ spectrum generating algebra can be performed as above. This corresponds to the right panel of figure 1. In conclusion, we have seen that both an $su(1,1)$ and an $su(2)$ algebra can be associated with the symmetric Pöschl–Teller potential, and their role depends on whether we consider the hyperbolic or the trigonometric version of this potential. It is notable that all these algebras can be obtained [33] as special cases from an algebraic description of the Ginocchio potential [9].

5. Summary

We have discussed the relation of supersymmetric quantum mechanics with two other symmetry concepts: $\mathcal{PT}$ symmetry and potential algebras. As all symmetries, these also have strong impact on the energy spectrum of quantum mechanical systems.

Combining the $\mathcal{PT}$ symmetry requirement with $N = 2$ SUSYQM led to relations between the real and imaginary components of the even and odd parts of the superpotential $W(x)$ expressed in terms of a system of inhomogeneous first-order differential equations. We presented methods to solve this system of differential equations and analysed the $\mathcal{PT}$ symmetry of the $V^{(+)}(x)$ SUSY partner potential. It was found that when the $\mathcal{PT}$ symmetry of $V^{(-)}(x)$ is unbroken, the system of differential equations reduces to a homogeneous form (as the factorization energy ε has to be real), and the trivial $W_{\mathrm{Re}} = W_{\mathrm{Io}}$ solutions apply, i.e. the superpotential has odd real and even imaginary components. In this case the $V^{(+)}(x)$ SUSY partner potential also exhibits $\mathcal{PT}$ symmetry. This was also demonstrated by the example of the $\mathcal{PT}$-symmetric Scarf II potential. When the $\mathcal{PT}$ symmetry of this potential is spontaneously broken, direct calculation showed that its SUSY partner ceases to be $\mathcal{PT}$-symmetric. One expects, however, that further solutions of the system of differential equations

formulated for the components of the superpotential might be possible. It would be interesting, for example, to find the solutions of the homogeneous system (real factorization energy) with a $V^{(+)}(x)$ potential that does not possess $\mathcal{PT}$ symmetry as in the case of the Scarf II potential. This would be indicative of the mechanism of the spontaneous breakdown of $\mathcal{PT}$ symmetry, which has been studied analytically only in the case of some solvable potentials [34–38]. The analysis of the components of the superpotential and of the SUSY partner potentials can also be helpful in finding relations between isospectral real and $\mathcal{PT}$-symmetric potentials.

We also analysed a particular realization of the non-compact $su(1, 1)$ algebra and its compact counterpart $su(2)$. We used first-order differential operators to construct potential and spectrum generating algebras. We showed that potential algebras are tightly related with the formalism of SUSYQM, as their J_+ and J_- ladder operators acted in the same way as the A and $A^\dagger$ operators of SUSYQM. We presented examples for both $su(1, 1)$ and $su(2)$ potential algebras, which connect infinite and finite number of degenerate levels, respectively. We used the hyperbolic and trigonometric versions of the Pöschl–Teller potential to demonstrate how the compact and non-compact potential algebras can be transformed into each other, and also presented $su(1, 1)$ and $su(2)$ spectrum generating algebras for the same systems.

The relation of $\mathcal{PT}$ symmetry and potential algebras has been discussed elsewhere [39, 40], and here we note only that in order to accommodate the two sets of states in $\mathcal{PT}$-symmetric potentials (distinguished by the q quasi-parity quantum number), the $su(1, 1) \sim so(2, 1)$ algebra has to be extended to $so(2, 2) \sim so(2, 1) \oplus so(2, 1)$. In terms of SUSYQM the duplication of potential algebras corresponds to the duplication of superpotentials ($W_q(x)$) and SUSY ladder operators (A_q and $A_q^\dagger$).

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References

[1] Cooper F, Khare A and Sukhatme U 1995 *Phys. Rep.* **251** 267
Junker G 1996 *Supersymmetric Methods in Quantum and Statistical Physics* (Berlin: Springer)
[2] Gendenshtein L E 1983 *Zh. Eksp. Teor. Fiz. Pisma. Red.* **38** 299
Gendenshtein L E 1983 *JETP Lett.* **38** 35 (Engl. Transl.)
[3] Lévai G 1989 *J. Phys. A: Math. Gen* **22** 689
[4] Infeld L and Hull T E 1951 *Rev. Mod. Phys.* **23** 21
[5] Miller W Jr 1968 *Lie Theory and Special Functions* (New York: Academic)
[6] Cooper F, Ginocchio J N and Khare A 1987 *Phys. Rev.* D **36** 2458
[7] Bhattacharjie A and Sudarshan E C G 1962 *Nuovo Cimento* **25** 864
[8] Natanzon G A 1971 *Teor. Mat. Fiz.* **38** 146
[9] Ginocchio J N 1984 *Ann. Phys., NY* **152** 203
[10] Ginocchio J N 1985 *Ann. Phys., NY* **159** 467
[11] Lévai G and Williams B W 1993 *J. Phys. A: Math. Gen.* **26** 3301
[12] Lévai G, Kónya B and Papp Z 1998 *J. Math. Phys.* **26** 5811
[13] Williams B W, Rutherford J L and Lévai G 1995 *Phys. Lett.* A **199** 7
[14] Williams B W and Lévai G 2003 *Mod. Phys. Lett.* A **18** 1901
[15] Bender C M and Boettcher S 1998 *Phys. Rev. Lett.* **80** 5243
[16] Mostafazadeh A 2002 *J. Math. Phys.* **43** 205
Mostafazadeh A 2002 *J. Math. Phys.* **43** 2814
Mostafazadeh A 2002 *J. Math. Phys.* **43** 3944
[17] Bender C M, Brody D C and Jones H F 2002 *Phys. Rev. Lett.* **89** 270401
[18] Lévai G and Znojil M 2000 *J. Phys. A: Math. Gen.* **33** 7165
[19] Lévai G and Znojil M 2001 *Mod. Phys. Lett.* A **30** 1973

[20] Bagchi B, Quesne C and Znojil M 2001 *Mod. Phys. Lett.* A **16** 2047
[21] Dorey P, Dunning C and Tateo R 2001 *J. Phys. A: Math. Gen.* **34** L391
[22] Andrianov A A, Ioffe M V, Cannata F and Dedonder J P 1999 *Int. J. Mod. Phys.* A **14** 2675
[23] Bagchi B, Mallik S and Quesne C 2002 *Mod. Phys. Lett.* A **17** 1651
[24] Lévai G and Znojil M 2002 *J. Phys. A: Math. Gen.* **35** 8793
[25] Kurzweil J 1986 *Ordinary Differential Equations; Studies in Applied Mathematics* vol 13 (Amsterdam: Elsevier)
[26] Bohm A, Ne'eman Y and Barut A O (ed) 1988 *Dynamical Groups and Spectrum Generating Algebras* (Singapore: World Scientific)
[27] Alhassid Y, Gürsey F and Iachello F 1986 *Ann. Phys., NY* **167** 181
[28] Wybourne B G 1974 *Classical Groups for Physicists* (New York: Wiley)
[29] Grudzinski K and Wybourne B G 1996 *Rep. Math. Phys.* **38** 251
[30] Lévai G 1994 *J. Phys. A: Math. Gen.* **27** 3809
[31] Balantekin A B 1998 *Phys. Rev.* A **57** 4188
[32] Gangopadhyaya A, Mallow J V and Sukhatme U P 1998 *Phys. Rev.* A **58** 4287
[33] Lévai G 1997 *Proc. 21st Int. Colloq. on Group Theoretical Methods in Physics* vol 1, ed H-D Doebner, P Nattermann and W Scherer (Singapore: World Scientific) p 461
[34] Ahmed Z 2001 *Phys. Lett.* A **282** 343
[35] Znojil M 1999 *Phys. Lett.* A **259** 220
[36] Lévai G, Cannata F and Ventura A 2002 *Phys. Lett.* A **300** 271
[37] Lévai G, Sinha A and Roy P 2003 *J. Phys. A: Math. Gen.* **36** 7611
[38] Sinha A, Lévai G and Roy P 2004 *Phys. Lett.* A **322** 78
[39] Lévai G, Cannata F and Ventura A 2001 *J. Phys. A: Math. Gen.* **34** 839
[40] Lévai G, Cannata F and Ventura A 2002 *J. Phys. A: Math. Gen.* **35** 5041